On simulation of Hamiltonians using local unitary transforamtions

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September,2001

Abstract

We give a necessary condition for the simulation of Hamiltonians, which is independent of the eigenvalues of Hamiltonians and based on the algebraic-geometric invariants recently introduced in [1] [2]. The result shows that the problem of simulation of Hamiltonians in arbitrary bipartite quantum systems cannot be described by only using eigenvalues, which is quite different to the two-qubit case.

Historically, the idea of simulating Hamiltonian time evolutions was the first motivation for quantum computation [3]. Recently the ability of nonlocal Hamiltonians to simulate one another is a popular topic , which has applications in quantum control theory [4], quantum compuation [5],[6],[7],[8] and the task of generating enatnglement [9] [10]. For the general treatments of this topic , we refer to [11],[12],[13].

We recall the definitions from [11] and [12].

Definition 1 ([11]). *Let H and H' be bipartite Hamiltonians on $H_A^m \otimes H_B^n$, H' can be efficiently simulated by H, write as $H' \prec_C H$, if the evolution according to $e^{-iH't'}$ for any time t' can be obtained by using the interaction H for the same period of time t' by manipulating $H_A^m \otimes H_B^n$ using the appropriate operations in the class C.*

In this paper, we are mainly interested in the case that the class C is the class of all local untary transformations (LU), ie., any operation in C is of the form $U_A \otimes U_B$, where U_A, U_B are the unitary transforamtions in H_A^m, H_B^n respectively.

From [11], it is known that actually the set $\{H' \prec_{LU} H\}$ is precissely the convex hull of the set $\{U \otimes VH(U^* \otimes V^*)^\tau\}$, this leads to the following definition of **first order simulation** in [12].

Definition 2 ([12]). *Let H and H' be bipartite Hamiltonians on $H_A^m \otimes H_B^n$, H' can be simulated by H with overhead 1, write as $H' \prec_C H$, if H' can be written as a convex combination of conjugates of H by elements in the class C, $H' = p_1 T_1 H (T_1^*)^\tau + ... + p_s T_s H (T_s^*)^\tau$, where $p_1, ..., p_s$ are positive real numbers such that $p_1 + .. + p_s = 1$ and $T_1, ..., T_s \in C$.*

When H is a positive self-adjoint operator, it is clear that H' has a to be a positive self-adjoint operator. If $H = |v >< v|$ and $H' = |v' >< v'|$ where $|v >$ and $|v' >$ are pure states and H' can be simulated by H efficiently, ie., $H' \prec_{LU} H$, actually the Schmidt ranks of $|v >$ and $|v' >$ have to be the same. In fact, if there exist positive numbers $p_1, ..., p_s$ and local unitary transformations $U_1 \otimes V_1, ..., U_s \otimes V_s$, such that, $\Sigma_i p_i U_i \otimes V_i H (U_i^* \otimes V_i^*)^\tau = H'$, it is clear that $U_i \otimes V_i H (U_i^* \otimes V_i^*)^\tau = |(U_i \otimes V_i)v >< (U_i \otimes V_i)v|$, and from the well-known fact in [14] Lemma 1, $|(U_i \otimes V_i)v >$ is in the range of H'. Hence $|v' >= |(U_i \otimes V_i)v >$ and the Schmidt ranks of $|v >$ and $|v' >$ have to be the same.

For positive self-adjoint operators H (equivalently, unnormalized mixed states) in bipartite quantum systems $H_A^m \otimes H_B^n$, algebriac sets $V_A^k(H)$ in CP^{m-1} (respectively $V_B^k(H)$ in CP^{n-1}) are introduced in [1] as the non-local invariants of H, ie., they are invariant when local unitary transformations applied to the positive Hamiltonians H. Moreover these algebraic sets are independent of the eigenvalues of H. From Proposition 1 in [1], Schmidt ranks of pure states are just the codimensions of the algebraic sets. Therefore it is natural to think the above observation about the equality of Schmidt ranks of rank 1 Hamiltonians which can be simulated efficiently can be extended to equality of these algebraic sets of arbitrary positive Hamiltonians if they can be simulated efficiently. In this paper we give such a necessary condition about the simulation of positive Hamiltonians based on these invariants.

Main Theorem. *Let H and H' be the positive Hamiltonians in the bipartite quantum system $H_A^m \otimes H_B^n$ with the same rank, ie., $dim(range(H)) = dim(range(H'))$. Suppose that $H' \prec_{LU} H$, that is , H' can be simulated by H efficiently by using local unitary transformations. Then $V_A^k(H) = V_A^k(H')$ for $k = 0, ..., n-1$ and $V_B^k(H) = V_B^k(H')$ for $k = 0, ..., m-1$, here equality of algebraic sets means they are isomorphic via projective linear transformations of complex projective spaces.*

Since the algebraic-geometric invariants are independent of eigenvalues, thus our above theorem is a necessary condition of simulation of Hamiltonians without refering to eigenvalues of Hamiltonians. On the other hand, recall the Theorem in section G of [11], for Hamiltonians $H = \Sigma_i h_i \sigma_i \otimes \sigma_i$ and $H' = \Sigma_i h'_i \sigma_i \otimes \sigma_i$, where σ_i's are Pauli matrices, on two-qubit systems, $H' \prec_{LU} H$ if and only if $h' \prec_s h$, where $\prec_s$ is the s-majoriaztion defined in [11]. Thus we can see that in the case of simualtion of Hamiltonians in two-qubit systems, eigenvalues of Hamiltonians plays a crucial role, since h and h' can be determined from the eigenvalues of Hamiltonians H and H' uniquely. However our above thereom implies that in the case of arbitrary bipartite quantum systems, the algebraic sets which are independent of eigenvaluse play a more fundamental role. This is also illustrated in the following example of simulation of Hamiltonians in $H_A^3 \otimes H_B^3$.

Example 1. Let $H = H_A^3 \otimes H_B^3$ be a bipartite quantum system and the following 3 unit vectors are in $H_A^3 \otimes H_B^3$.

$$|v_1>=\frac{1}{\sqrt{3}}(e^{i\eta_1}|11>+|22>+|33>)$$
$$|v_2>=\frac{1}{\sqrt{3}}(e^{i\eta_2}|12>+|23>+|31>) \quad (1)$$
$$|v_3>=\frac{1}{\sqrt{3}}(e^{i\eta_3}|13>+|21>+|32>)$$

,where η_1,η_2,η_3 are 3 real parameters. Let $H_{\eta_1,\eta_2,\eta_3}=(|v_1><v_1|+|v_2><v_2|+|v_3><v_3|$. This is a continuous family of Hamiltonians in H_{η_1,η_2,η_3} of rank 3 parameterized by three real parameters.

As in [1], $V_A^2(H_{\eta_1,\eta_2,\eta_3})$ is just the elliptic curve in CP^2 defined by $r_1^3+r_2^3+r_3^3-\frac{e^{i\eta_1}+e^{i\eta_2}+e^{i\eta_3}}{e^{i(\eta_1+\eta_2+\eta_3)/3}}r_1r_2r_3=0$. Set $g(\eta_1,\eta_2,\eta_3)=\frac{e^{i\eta_1}+e^{i\eta_2}+e^{i\eta_3}}{e^{i(\eta_1+\eta_2+\eta_3)/3}}$, we know from the well-known fact in algebraic geometry that the elliptic curve $V_A^2(H_{\eta_1,\eta_2,\eta_3})$ is not isomorphic to the elliptic curve $V_A^2(H_{\eta_1',\eta_2',\eta_3'})$ if $k(g(\eta_1,\eta_2,\eta_3))\neq k(g(\eta_1',\eta_2',\eta_3'))$, where $k(x)=\frac{x^3(x^3+216)^3}{(-x^3+27)^3}$ is the moduli function of elliptic curves (see [15]).

From the main Theorem we have the following result.

Corollary 1. *$H_{\eta_1',\eta_2',\eta_3'}$ cannot be simulated by H_{η_1,η_2,η_3} efficiently by using local unitary transformations,ie.,we cannot have $H_{\eta_1',\eta_2',\eta_3'}\prec_{LU}H_{\eta_1,\eta_2,\eta_3}$, if $k(g(\eta_1,\eta_2,\eta_3))\neq k(g(\eta_1',\eta_2',\eta_3'))$, though the 3 nonzero eigenvalues of H_{η_1,η_2,η_3}, $H_{\eta_1',\eta_2',\eta_3'}$ and their partial traces are all 1.*

Proof. It is easy to calculate the eigenvalues to check the 2nd conclusion. The first conclusion is from main theorem and the well-known fact about elliptic curves in [15] mentioned above.

For example, for the 2 Hamiltonians $H=|\psi_1><\psi_1|+|\psi_2><\psi_2|+|\psi_3><\psi_3|$ and $H'=|\psi_1'><\psi_1'|+|\psi_2'><\psi_2'|+|\psi_3'><\psi_3'|$, where

$$\begin{array}{l}
|\psi_1>=\frac{1}{\sqrt{3}}(|11>+|22>+|33>)\\
|\psi_2>=\frac{1}{\sqrt{3}}(|12>+|23>+|31>)\\
|\psi_3>=\frac{1}{\sqrt{3}}(|13>+|21>+|32>)\\
|\psi_1'>=\frac{1}{\sqrt{3}}(|11>+|22>+|33>)\\
|\psi_2'>=\frac{1}{\sqrt{3}}(|12>+|23>+|31>)\\
|\psi_3'>=\frac{1}{\sqrt{3}}(-|13>+|21>+|32>)
\end{array} \quad (2)$$

we know from Corollary 1 that H' cannot be simulated by H efficintly, ie., we cannot have $H' \prec_{LU} H$, from the simple calculation of the moduli function in Corollary 1.

This example strongly suggests that the problem of simulation of Hamiltonians in arbitrary bipartite quantum systems is quite different to the problem in two-qubit case as studied in [11].

For the proof of the main theorem, we first recall the definition of algebraic sets of positive self-adjoint operators in bipartite quantum systems and how to compute them in [1].

Let $H_A^m \otimes H_B^n$ be a bipartite system and the standard orthogonal base is $\{|ij>\}$, where, $i = 1, ..., m$ and $j = 1, ..., n$, and ρ is an arbitrary positive self-adjoint operator. We represent the matrix of ρ in the base $\{|11>, ...|1n>, ..., |m1>, ..., |mn>\}$, and consider ρ as a blocked matrix $\rho = (\rho_{ij})_{1 \leq i \leq m, 1 \leq j \leq m}$ with each block ρ_{ij} a $n \times n$ matrix corresponding to the $|i1>, ..., |in>$ rows and the $|j1>, ..., |jn>$ columns.We define

$$V_A^k(\rho) = \{(r_1, ..., r_m) \in CP^{m-1} : rank(\Sigma_{i,j} r_i r_j^* \rho_{ij}) \leq k\} \quad (3)$$

for $k = 0, 1, ..., n-1$. Similarly $V_B^k(\rho) \subseteq CP^{n-1}$ can be defined. Here * means the conjugate of complex numbers. It is known from Theorem 1 and 2 of [1] that these sets are algebraic sets (zero locus of several multi-variable polynomials, see [16]) and they are invariants under local unitary transformations. Actually these algebraic sets can be computed easily as follows.

Let $\{|11>, ..., |1n>, ..., |m1>, ..., |mn>\}$ be the standard orthogonal base of $H_A^m \otimes H_B^n$ as above and $\rho = \Sigma_{l=1}^t p_l |v_l><v_l|$ be any given representation of ρ as a convex combination of projections with $p_1, ..., p_t > 0$. Suppose $v_l = \Sigma_{i,j=1}^{m,n} a_{ijl} |ij>$, $A = (a_{ijl})_{1 \leq i \leq m, 1 \leq j \leq n, 1 \leq l \leq t}$ is the $mn \times t$ matrix. Then it is clear that the matrix representation of ρ with the base $\{|11>, ..., |1n>, ..., |m1>, ..., |mn>\}$ is $AP(A^*)^\tau$, where P is the diagonal matrix with diagonal entries $p_1, ..., p_t$. We may consider the $mn \times t$ matrix A as a $m \times 1$ blocked matrix with each block A_w, where $w = 1, ..., m$, a $n \times t$ matrix corresponding to $\{|w1>, ..., |wn>\}$. Then $V_A^k(\rho)$ is just the algebraic set in CP^{m-1} as the zero locus of the determinants of all $(k+1) \times (k+1)$

submatrices of $\Sigma_i^m r_i A_i$.

The following obsevation is the the key point of the proof of main theorem. From Lemma 1 in [14], the range of ρ is the linear span of vectors $|v_1>, ..., |v_t>$. We take any $dim(range(\rho))$ linear independent vectors in the set $\{|v_1>, ..., |v_t>\}$, say they are $|v_1>, ..., |v_s>$, where $s = dim(range(\rho))$. Let B be the $mn \times s$ matrix with columns corresponding to the s vectors $|v_1>, ..., |v_s>$'s coordinates in the standard base of $H_A^m \otimes H_B^n$. We consider B as $m \times 1$ blocked matrix with blocks $B_1, ..., B_m$ $n \times s$ matrix as above. It is clear that $V_A^k(\rho)$ is just the zero locus of determinants of all $(k+1) \times (k+1)$ submatrices of $\Sigma_i^m r_i B_i$, since any column in $\Sigma_i r_i A_i$ is a linear combination of columns in $\Sigma_i r_i B_i$.

Proof of main theorem. Suppose $H' \prec_{LU} H$, then there exist positive numbers $p_1, ..., p_s$ and local unitary transformations $U_1 \otimes V_1, ..., U_s \otimes V_s$, such that, $\Sigma_i p_i U_i \otimes V_i H (U_i^* \otimes V_i^*)^\tau = H'$. Let $H = \Sigma_i^s q_i |\psi_i><\psi_i|$, where $s = dim(range(H))$, $q_1, ..., q_s$ are eigenvalues of H and $|\psi_1>, ..., |\psi_s>$ are eigenvectors of H. Then it is clear that $(U_i \otimes V_i) H (U_i^* \otimes V_i^*)^\tau = \Sigma_j^s q_j |(U_i \otimes V_i)\psi_j><(U_i \otimes V_i)\psi_j|$ and thus $H' = \Sigma_{i,j} p_i q_j |(U_i \otimes V_i)\psi_j><(U_i \otimes V_i)\psi_j|$. This is a representation of H' as a convex combination of projections. From our above observation $V_A^k(H')$ can be computed from vectors $|(U_1 \otimes V_1)\psi_1>, ..., |(U_1 \otimes V_1)\psi_s>$, since they are linear independent and $s = dim(range(H'))$. Hence $V_A^k(H') = V_A^k((U_1 \otimes V_1)H)$ from the definition. Thus the conclusion follows from Theorem 1 in [1].

The main theorem even plays certain role in simulation of two-qubit Hamiltonians as illustrated in the following example.

Example 2. Let $H = \lambda_1 |\psi_1><\psi_1| + \lambda_2 |\psi_2><\psi_2|$ and $H' = \lambda_1' |\psi_1'><\psi_1'| + \lambda_2' |\psi_2'><\psi_2'|$ be two Hamiltonians on $H_A^2 \otimes H_B^2$, where λ's are any given positive real numbers and

$$\begin{array}{c} |\psi_1>=|11> \\ |\psi_2>=|22> \\ |\psi_1'>=\frac{1}{\sqrt{2}}(|11>+|22>) \\ |\psi_2'>=|12> \end{array} \quad (4)$$

Then we know H and H' are two rank 2 Hamiltonians. It is easy to compute that $V_A^1(H)$ is the algebraic set of two points $(1:0)$ and $(0:1)$ in CP^1 and $V_A^1(H')$ is the algebraic set of one point $(0:1)$ in CP^1. Hence we cannot have $H' \prec_{LU} H$ from the main theorem.

The following example is from the main theorem and well-known facts about rational normal scrolls (see [16]).

Example 3. Let $\phi_1 = \frac{1}{\sqrt{2}}(|01> + |12>)$,..,$\phi_i = \frac{1}{\sqrt{2}}(|i-1,1> + |i2>)$,...,$\phi_l = \frac{1}{\sqrt{2}}(|l-1,1> + |l,2>)$,$\phi_{l+1} = \frac{1}{\sqrt{2}}(|l+1,1> + |l+2,2>)$,...,$\phi_j = \frac{1}{\sqrt{2}}(|j1> + |j+1,2>)$,...,$\phi_{n-1} = \frac{1}{\sqrt{2}}(|n-1,1> + |n2>)$ be vectors in $H_A^{n+1} \otimes H_B^2$. We consider the Hamiltonians $H_l = \frac{1}{n-1}(|\phi_1><\phi_1| + ... + |\phi_{n-1}><\phi_{n-1}|)$ of rank $n-1$ for $l = 1, ..., [\frac{n-1}{2}]$. It is clear from [1] $V_A^1(H_l) = X_{l,n-l-1} \subset CP^n$, the rational normal scroll (p.106, [16]). From the well-known fact in algebraic geometry (see pp.92-93 and p.106 of [16]) we have the following result.

Corollary 2. *We have $H_{l'} \prec_{LU} H_l$ for $l, l' = 1, ..., [\frac{n-1}{2}]$ if and only if $l = l'$.*

Proof. From main theorem $H_{l'} \prec_{LU} H_l$ implies $X_{l,n-l-1}$ and $X_{l',n-l'-1}$ are isomorphic by a projective linear transformaton of CP^n. Thus the conclusion follows from Proposition 8.20 of [16].

Actually the algebraic geometric invariants in [1] can be used to give a more general necessary condition for the simulation of Hamiltonians by using local untary transformations.

Theorem 2. *Let H and H' be two positive Hamitonians on $H_A^m \otimes H_B^n$. Suppose that there exists a representation of H as a convex combination $H = \Sigma_i^t q_i|v_i><v_i|$, with positive q_i's and the Schmidt rank of $|v_1>$ is $min\{m,n\}$. Moreover $V_A^0(H')$ is not empty. Then H' cannot be simulated by H efficiently by using local unitary transformations, ie., we cannot have $H' \prec_{LU} H$.*

Proof. From the condition, there exist positive $p_1, ..., p_s$ and local unitary transformations $U_1\otimes V_1,...,U_s\otimes V_s$, such that, $\Sigma_i^s p_i U_i\otimes V_i H(U_i^*\otimes V_i^*)^\tau = H'$. It is clear that $(U_i\otimes V_i)H(U_i^*\otimes V_i^*)^\tau = \Sigma_j^t q_j|(U_i\otimes V_i)v_j >< (U_i\otimes V_i)v_j|$, and thus $H' = \Sigma_{i,j}^{s,t} p_i q_j|(U_i\otimes V_i)v_j >< (U_i\otimes V_i)v_j|$. From Lemma 1 in [14], $range(H')$ is the linear span of vectors $(U_i\otimes V_i)v_j$ for $i=1,...,s$ and $j=1,...,t$. From the above description about the computation of $V_A^0(H')$, we can compute it by choosing $dim(range(H'))$ linear independent vectors in this set $\{(U_1\otimes V_1)v_1,...,(U_1\otimes V_1)v_t,...,(U_s\otimes V_s)v_1,...,(U_s\otimes V_s)v_t\}$. Therefore we can choose one of these $dim(range(H'))$ linear independent vectors to be $(U_1\otimes V_1)v_1$, whose Schmidt rank is $min\{m,n\}$. From Proposition 1 in [1] and the definition , thus we know that $V_A^0(H')$ has to be the empty set. This is a contradiction and the conclusion is proved.

Example 4. Let $H = |v >< v|$ and $H' = |u_1 >< u_1| + |u_2 >< u_2|$ be two Hamiltonians on $H_A^3\otimes H_B^3$ where

$$\begin{array}{c} v = \frac{1}{2}(|11> + |22> + |33>) \\ u_1 = \frac{1}{\sqrt{2}}(|11> + |22>) \\ u_2 = \frac{1}{\sqrt{2}}(|11> - |22>) \end{array} \quad (5)$$

It is clear that $V_A^0(H')$ is the set of all points $(0:0:1)$ in CP^3, thus nonempty. On the other hand H satisfies the condition in Theorem 2. Thus we cannot have $H' \prec_{LU} H$.

Example 5. Let $H = |v_1 >< v_1| + |v_2 >< v_2|$ and $H' = |u_1 >< u_1| + |u_2 >< u_2| + |u_3 >< u_3|$ be two Hamiltonians on $H_A^5\otimes H_B^5$ where

$$\begin{array}{c} v_1 = \frac{1}{\sqrt{5}}(|11> + |22> + |33> + |44> + |55>) \\ v_2 = |12> \\ u_1 = \frac{1}{\sqrt{6}}(|13> + |14> + |15> + |23> + |24> + |25>) \\ u_2 = \frac{1}{2}(|31> + |41> + |32> + |42>) \\ u_3 = |55> \end{array} \quad (6)$$

It is clear that $V_A^0(H')$ is the set defined by $r_1 + r_2 = 0$, $r_3 + r_4 = 0$ and $r_5 = 0$ in CP^4, thus a dimension 1 linear subspace and nonempty. On the other hand H satisfies the condition in Theorem 2. Thus we cannot have

$H' \prec_{LU} H$.

The author acknowledges the support from NNSF China, Information Science Division, grant 69972049.

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